Topology and Its Impact on Superconducting Quantum Networks

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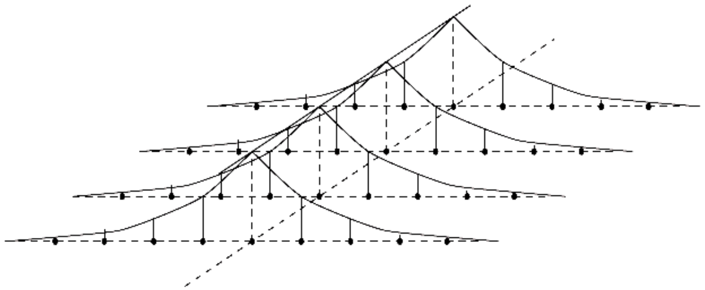
Outlines

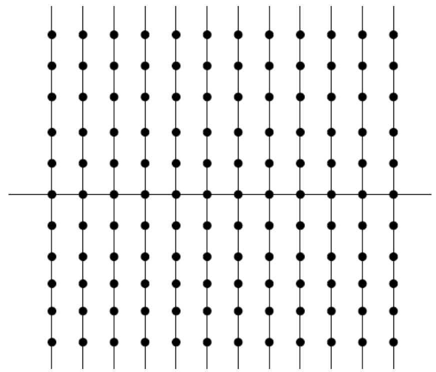
- ❑ Motivations
- ✓ From the Burioni's original proposal to the experimental realization
- ❑ Cooper pairs localization in quantum graphs: A generalized Feynman model
- ❑A GL formulation for the superconductive transition in quantum graphs: Critical temperature enhancement
- ❑ Searching for a microscopic theory: A real space BCS model (rsBCS)
- ✓ Two- and few- body problem in quantum graphs within the rsBCS model
- ❑ Qubits, topology, and…
- ❑Conclusions and perspectives



Motivations

EUROPHYSICS LETTERS 1 November 2000

Europhys. Lett., **52** (3), pp. 251–256 (2000)



Bose-Einstein condensation in inhomogeneous Josephson arrays

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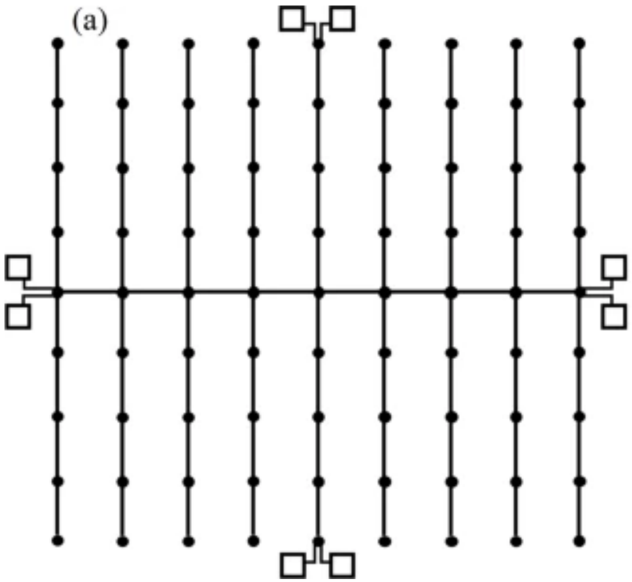
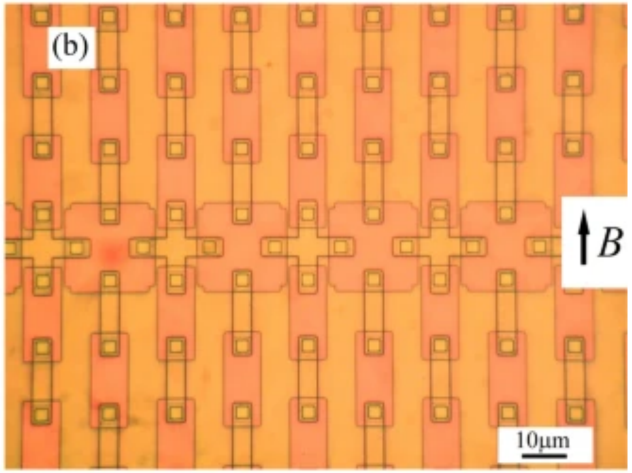
PACS. 03.75.Fi – Phase coherent atomic ensembles; quantum condensation phenomena.
PACS. 85.25.Cp – Josephson devices.
PACS. 74.80.-g – Spatially inhomogeneous structures.

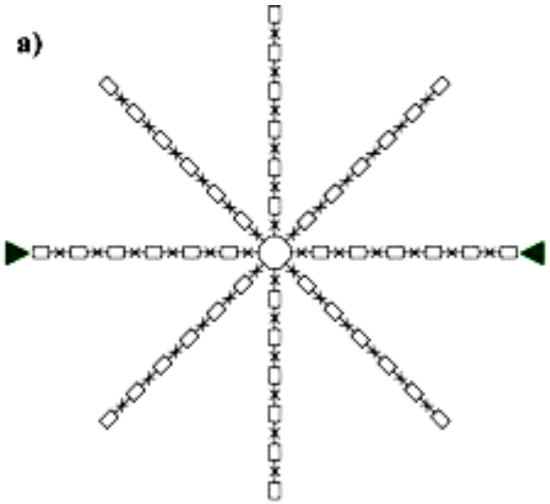
Abstract. – We show that spatial Bose-Einstein condensation of non-interacting bosons occurs in dimension $d < 2$ over discrete structures with inhomogeneous topology and with no need of external confining potentials. Josephson junction arrays provide a physical realization of this mechanism. The topological origin of the phenomenon may open the way to the engineering of quantum devices based on Bose-Einstein condensation. The comb array, which embodies all the relevant features of this effect, is studied in detail.

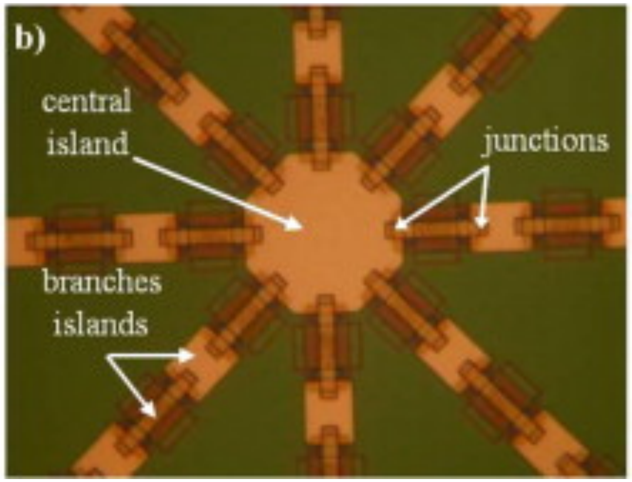
Combs and Stars





M. Lucci et al., Sci. Rep. **10**, 10222 (2020)





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Experimental realizations

- P. Silvestrini, R. Russo, V. Corato, B. Ruggiero, C. Granata, S. Rombetto, M. Russo, M. Cirillo, A. Trombettoni, P. Sodano, Topology-induced critical current enhancement in Josephson networks. Phys. Lett. A **370**, 499–503 (2007)
- M. Lorenzo, M. Lucci, V. Merlo, I. Ottaviani, M. Salvato, M. Cirillo, F. Müller, T. Weimann, M.G. Castellano, F. Chiarello, G. Torrioli, On Bose-Einstein condensation in Josephson junctions star graph arrays. Phys. Lett. A **378**, 655–658 (2014)
- I. Ottaviani, M. Lucci, R. Menditto, V. Merlo, M. Salvato, M. Cirillo, F. Müller, T. Weimann, M.G. Castellano, F. Chiarello, G. Torrioli, R. Russo, Characterization of anomalous pair currents in Josephson junction networks. J. Phys. Condens. Matter **26**, 215701 (2014)
- M. Lucci, D. Cassi, V. Merlo, R. Russo, G. Salina, M. Cirillo, Conditioning of superconductive properties in graph-shaped reticles. Sci. Rep. **10**, 10222 (2020)
- M. Lucci, D. Cassi, V. Merlo, R. Russo, G. Salina, M. Cirillo, Josephson currents and gap enhancement in graph arrays of superconductive islands. Entropy **23**(7), 811 (2021)
- M. Lucci, V. Campanari, D. Cassi, V. Merlo, F. Romeo, G. Salina, M. Cirillo, Quantum coherence in loopless superconductive networks. Entropy **24**(11), 1690 (2022)

Theoretical works discussed in this talk

- [Ref.1] Romeo, F., De Luca, R. Cooper pairs localization in tree-like networks of superconducting islands. Eur. Phys. J. Plus **137**, 726 (2022).
- [Ref.2] Romeo, F. Order parameter focalization and critical temperature enhancement in synthetic networks of superconducting islands. J. Phys.: Condens. Matter **33,** 045401 (2021).
- [Ref.3] Romeo, F. On the Bardeen–Cooper–Schrieffer interaction in quantum graphs. Eur. Phys. J. Plus **138**, 463 (2023).

Cooper pairs localization in quantum graphs: A generalized Feynman model [Ref.1]

Bose-Hubbard model & quantum graphs

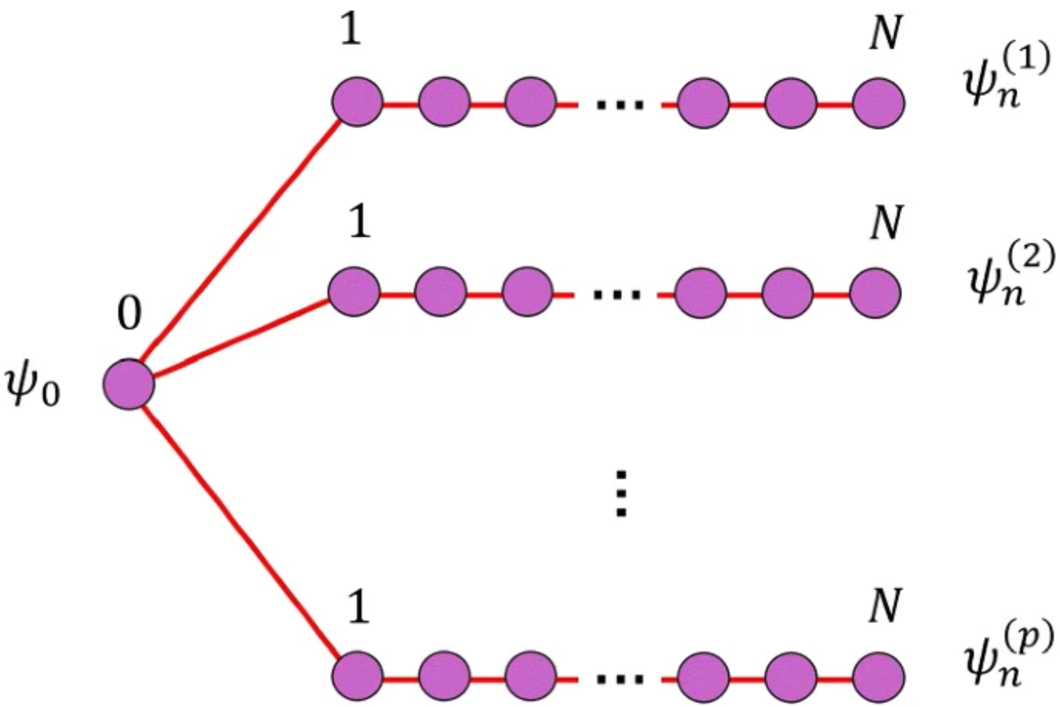
$$H = \sum_i \epsilon_i n_i - K \sum_{ij} \mathcal{A}_{ij} b_i^\dagger b_j + \frac{U}{2} \sum_i n_i (n_i - 1)$$

Under macroscopic occupation of the islands, we obtain the non-linear Schrodinger equation:

$$i\hbar \frac{d}{dt} \psi_k = \epsilon_k \psi_k - K \sum_j \mathcal{A}_{kj} \psi_j + U |\psi_k|^2 \psi_k$$

which can be seen, by neglecting the non-linear term, as a generalization of the Feynman model to the case of a quantum graph.

Star-like networks (U=0)



- Bosons localize on the center of the star.
- Moderate repulsion does not affect the result too much.
- Connectivity plays the role of a rather strong attractive onsite potential.

Main results

$$\frac{N_0}{N_T} = \frac{1}{2}\left(\frac{p-2}{p-1}\right)$$

$$N_n^{(j)} = N_0 \rho^{2n}$$

$$\rho = \frac{1}{\sqrt{p-1}}$$

A GL formulation for the superconductive transition in quantum graphs [Ref.2]

$$\mathcal{F} = \sum_{ij} \psi_i^* H_{ij} \psi_j + \sum_i \mathcal{A}(T) |\psi_i|^2 + \sum_i \mathcal{B} |\psi_i|^4$$

Free Energy functional for a quantum graph

$$\frac{\partial \mathcal{F}_\ell}{\partial \psi_k^*} = \sum_j H_{kj} \psi_j + \mathcal{A}(T)\, \psi_k = 0$$

Minimization in close vicinity of the transition temperature

$$H\psi = -\mathcal{A}(T)\, \psi$$

An eigenvalues problem…

$$\mathcal{T}_c = T_c \left(1 - \frac{\varepsilon_0}{\alpha T_c} \right)$$

Network transition temperature can be obtained

Order parameter focalization in star graphs

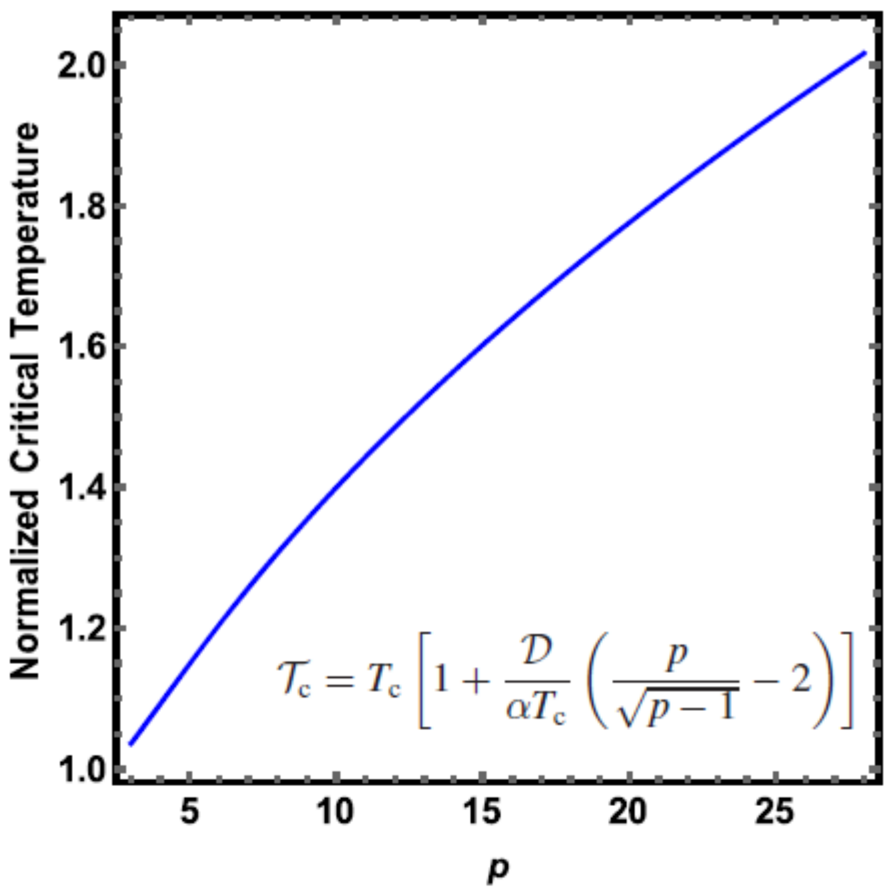
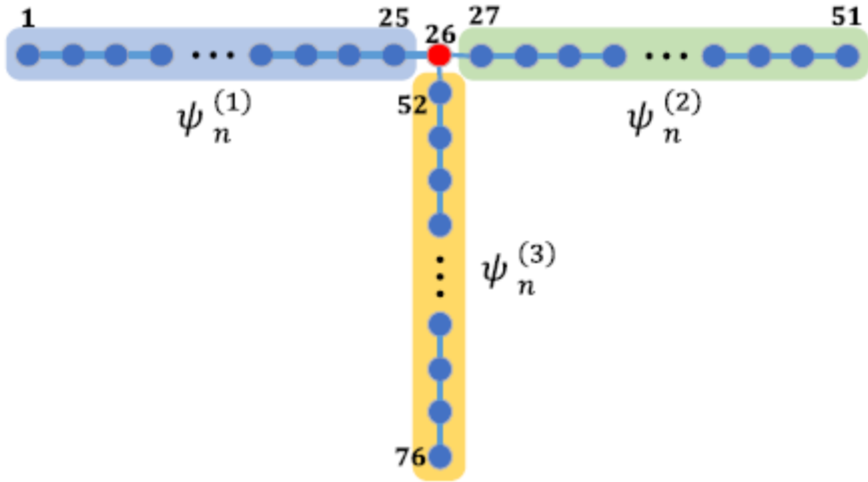


Figure 4. Normalized critical temperature $\mathcal{T}_c/T_c$ as a function of the number of legs p obtained by using equation (27) with $\mathcal{D}(\alpha T_c)^{-1} = 0.3$.



- **Enhancement** of the network transition temperature

Order parameter profile at the transition temperatur e

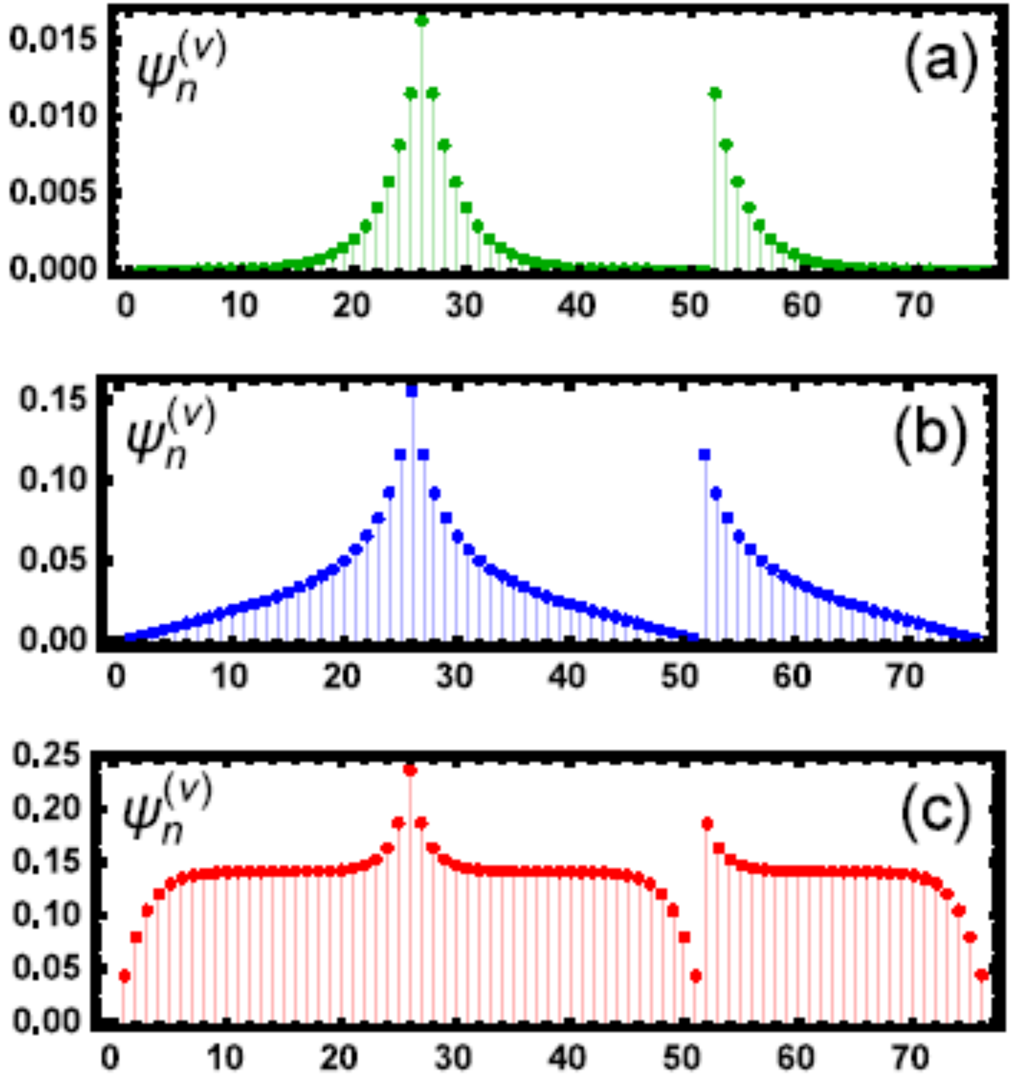
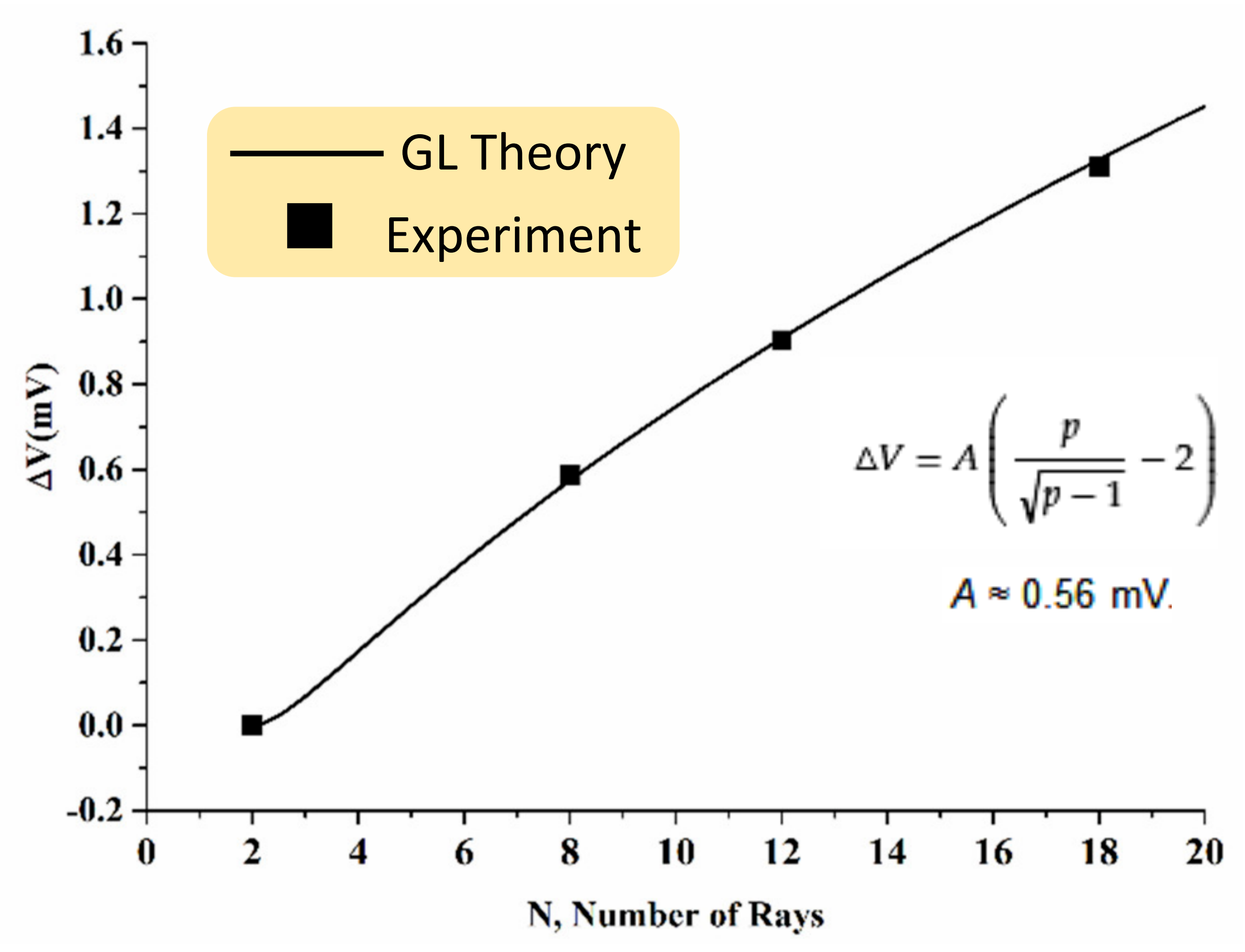
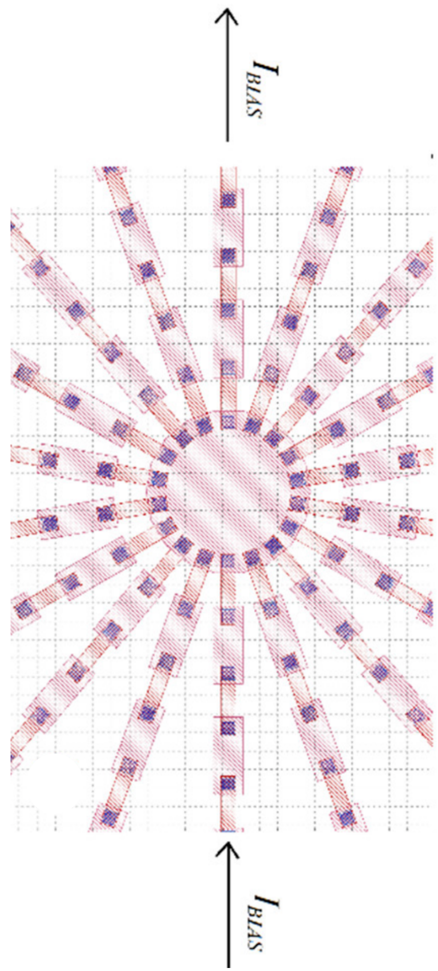


Figure 6. Order parameter $\psi_n^{(v)}$ for three legs star graph ($p = 3$) shown in figure 5(a). The order parameter profile has been computed by taking into account nonlinear terms and by fixing $\mathcal{D}(\alpha T_c)^{-1} = 0.1$. The system critical temperature is $\mathcal{T}_c/T_c \approx 1.01213$. Different panels have been obtained by fixing different temperature values. (a) $T = 1.012T_c$; (b) $T = 1.0T_c$; (c) $T = 0.98T_c$.

Gap voltage-sum scaling





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Intermediate conclusions

- Critical temperature of a synthetic superconductor can be enhanced by reducing the effective coherence length of the system. A similar mechanism works, under appropriate circumstances, in disordered superconductors (see M.N. Gastiasoro, B.M. Andersen, Enhancing superconductivity by disorder. Phys. Rev. B **98**, 184510 (2018), for details).
- The latter mechanism is related to the confining properties of network nodes with high connectivity.
- Critical temperature enhancement in synthetic networks suggests that confinement induced by the network topology is able to enhance the effective electron-electron attraction.
- In order to test the validity of the above statement a real space formulation of the BCS interaction is needed.

Is there already an available real-space BCS theory?

From De Gennes book "Superconductivity in metals and alloys"

Attractive Hubbard model

5

THE SELF-CONSISTENT FIELD METHOD

5–1 THE BOGOLUBOV EQUATIONS

We now describe a more powerful method (Bogolubov, 1959), which is essentially a generalization of the Hartree-Fock equations to the case of superconductivity. We begin by rewriting the Hamiltonian $\mathcal{H}$ of the electron system, not with the operators $a_{k\alpha}$ (which are only

($\alpha = \uparrow$ or $\downarrow$ is again a spin index.) The operators Ψ satisfy the anti-commutation rules

The Hamiltonian $\mathcal{H}$ is also very simply written in terms of Ψ and Ψ^+.[1]

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{5-7}$$

$$\mathcal{H}_0 = \int d\mathbf{r} \sum_{\alpha} \Psi^+(\mathbf{r}\alpha) \left[\frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{2m} + U_0(\mathbf{r}) \right] \Psi(\mathbf{r}) \tag{5-8}$$

$$\mathcal{H}_1 = -\tfrac{1}{2} V \int d\mathbf{r} \sum_{\alpha\beta} \Psi^+(\mathbf{r}\alpha)\Psi^+(\mathbf{r}\beta)\Psi(\mathbf{r}\beta)\Psi(\mathbf{r}\alpha) \tag{5-9}$$

[1]See Landau and Lifschitz, *Nonrelativistic Quantum Mechanics* (New York: Pergamon, 1959), Chap. 9.

Is there already an available real-space BCS theory?

Definition of the Effective Potentials

We now replace the interaction $V\Psi^+\Psi^+\Psi\Psi$ by an average potential acting on only one particle at a time (therefore only containing two operators Ψ or Ψ^+). We try an effective hamiltonian of the form

$$\mathcal{H}_{\text{eff}} = \int d\mathbf{r}\,\Big\{\sum_\alpha \Psi^+(\mathbf{r}\alpha)\mathcal{H}_e(\mathbf{r})\Psi(\mathbf{r}\alpha) + U(\mathbf{r})\Psi^+(\mathbf{r}\alpha)\Psi(\mathbf{r}\alpha)$$
$$+ \Delta(\mathbf{r})\Psi^+(\mathbf{r}\uparrow)\Psi^+(\mathbf{r}\downarrow) + \Delta^*(\mathbf{r})\Psi(\mathbf{r}\downarrow)\Psi(\mathbf{r}\uparrow)\Big\} \qquad (5\text{-}12)$$

...

$$[\Psi(\mathbf{r}\uparrow), \mathcal{H}_{\text{eff}}] = [\mathcal{H}_e + U(\mathbf{r})]\Psi(\mathbf{r}\uparrow) + \Delta(\mathbf{r})\Psi^+(\mathbf{r}\downarrow)$$
$$[\Psi(\mathbf{r}\downarrow), \mathcal{H}_{\text{eff}}] = [\mathcal{H}_e + U(\mathbf{r})]\Psi(\mathbf{r}\downarrow) - \Delta^*(\mathbf{r})\Psi^+(\mathbf{r}\uparrow) \qquad (5\text{-}17)$$

In this equality, we replace the Ψ's by the γ's by means of (5-13). Then we apply the commutation relations (5-16). Comparing the coefficients of γ_n (and γ_n^+) on the two sides of the equation, we obtain the *Bogolubov equations*:

$$\epsilon u(\mathbf{r}) = [\mathcal{H}_e + U(\mathbf{r})]u(\mathbf{r}) + \Delta(\mathbf{r})v(\mathbf{r})$$
$$\epsilon v(\mathbf{r}) = -[\mathcal{H}_e^* + U(\mathbf{r})]v(\mathbf{r}) + \Delta^*(\mathbf{r})u(\mathbf{r}) \qquad (5\text{-}18)$$

Is this a real-space BCS model?

Following a different path… [Ref.3]

Let us start with the Hamiltonian of a translational-invariant system in one dimension. In momentum space, interaction between fermions is described by the Hamiltonian:

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} - g \sum_{kk'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

Now, we assume that the real space version of the kinetic term is given by:

$$H_0 = -K \sum_{j\sigma} c^\dagger_{j+1\sigma} c_{j\sigma} + h.c.$$

Using non interacting fermionic fields, we obtain a real space model for the BCS interaction:

$$H = H_0 - g \sum_{\ell r} c^\dagger_{\ell\uparrow} c^\dagger_{\ell\downarrow} c_{r\downarrow} c_{r\uparrow}$$

Now, we assume that the interaction model is also valid when translational invariance is broken. Thus, a rsBCS model suitable to describe quantum graphs takes the form:

$$\mathcal{H} = \sum_{ij\sigma} c_{i\sigma}^{\dagger} h_{ij} c_{j\sigma} - g \sum_{\ell r} c_{\ell\uparrow}^{\dagger} c_{\ell\downarrow}^{\dagger} c_{r\downarrow} c_{r\uparrow}$$

Interestingly, the interaction term can be written in the general form:

$$H_I = -g \sum_{\ell r} \Gamma_{\ell r} c_{\ell\uparrow}^{\dagger} c_{\ell\downarrow}^{\dagger} c_{r\downarrow} c_{r\uparrow}$$

where:

$\Gamma_{\ell r} = 1$ rsBCS model

$\Gamma_{\ell r} = \delta_{\ell r}$ Attractive Hubbard model

$\Gamma_{\ell r} = \cdots$ Penson-Kolb pair hopping

Richardson's integrability of the rsBCS model

$$\mathcal{H} = \sum_{ij\sigma} c^{\dagger}_{i\sigma} h_{ij} c_{j\sigma} - g \sum_{\ell r} c^{\dagger}_{\ell\uparrow} c^{\dagger}_{\ell\downarrow} c_{r\downarrow} c_{r\uparrow}$$

Hereafter, we demonstrate that the many-body problem admits an exact solution in terms of Richardson's ansatz. Thus, the Hamiltonian is rewritten in terms of fermionic fields $a{\downarrow}i\sigma$ diagonalizing the non-interacting part of the Hamiltonian. Using the unitary transformation

$$c_{i\sigma} = \sum_j U_{ij} a_{j\sigma}$$

we obtain:

$$\mathcal{H} = \sum_{i\sigma} E_i a^{\dagger}_{i\sigma} a_{i\sigma} - g \sum_{ijkl} V^*_{ij} V_{kl} a^{\dagger}_{i\uparrow} a^{\dagger}_{j\downarrow} a_{k\downarrow} a_{l\uparrow}$$

with:

$$V_{ij} = \sum_k U_{ki} U_{kj}$$

Solving the many-body problem appears to be hopeless. However, when $h\downarrow ij$ is a real and symmetric matrix, the single-particle problem admits real-valued eigenstates. Under this condition, we obtain:

$$U_{ij} = U_{ij}^*$$

$$\delta_{ij} = \sum_k U_{ki}^* U_{kj} = \sum_k U_{ki} U_{kj} = V_{ij}$$

so that:

$$\mathcal{H} = \sum_{i\sigma} E_i a_{i\sigma}^\dagger a_{i\sigma} - g \sum_{ij} a_{i\uparrow}^\dagger a_{i\downarrow}^\dagger a_{j\downarrow} a_{j\uparrow}$$

Surprisingly, the Hamiltonian model written above, already known in the literature as *reduced BCS model*, is used to describe ultrasmall superconducting grains and the related many-body problem admits exact solution as shown by Richardson.

Two-body problem with translational invariance (Hubbard case)

We study two fermions in a singlet state interacting on a one-dimensional lattice. For Hubbard interaction, the orbital part of the two-body wavefunction is solution of the stationary Schrodinger equation:

$$(\Delta_1 + \Delta_2)\phi(x_1, x_2) - g\delta_{x_1 x_2}\phi(x_1, x_2) = E\phi(x_1, x_2)$$

It is easy to show that paired states form a narrow band with energy dispersion given by:

$$E_p = -\sqrt{g^2 + 16K^2\cos^2(p)}$$

where p is related to the non-vanishing center of mass momentum.

Two-body problem with translational invariance (rsBCS model)

An analogous problem can be solved in the case of the rsBCS interaction. In the latter case, the Schrödinger equation takes the peculiar form:

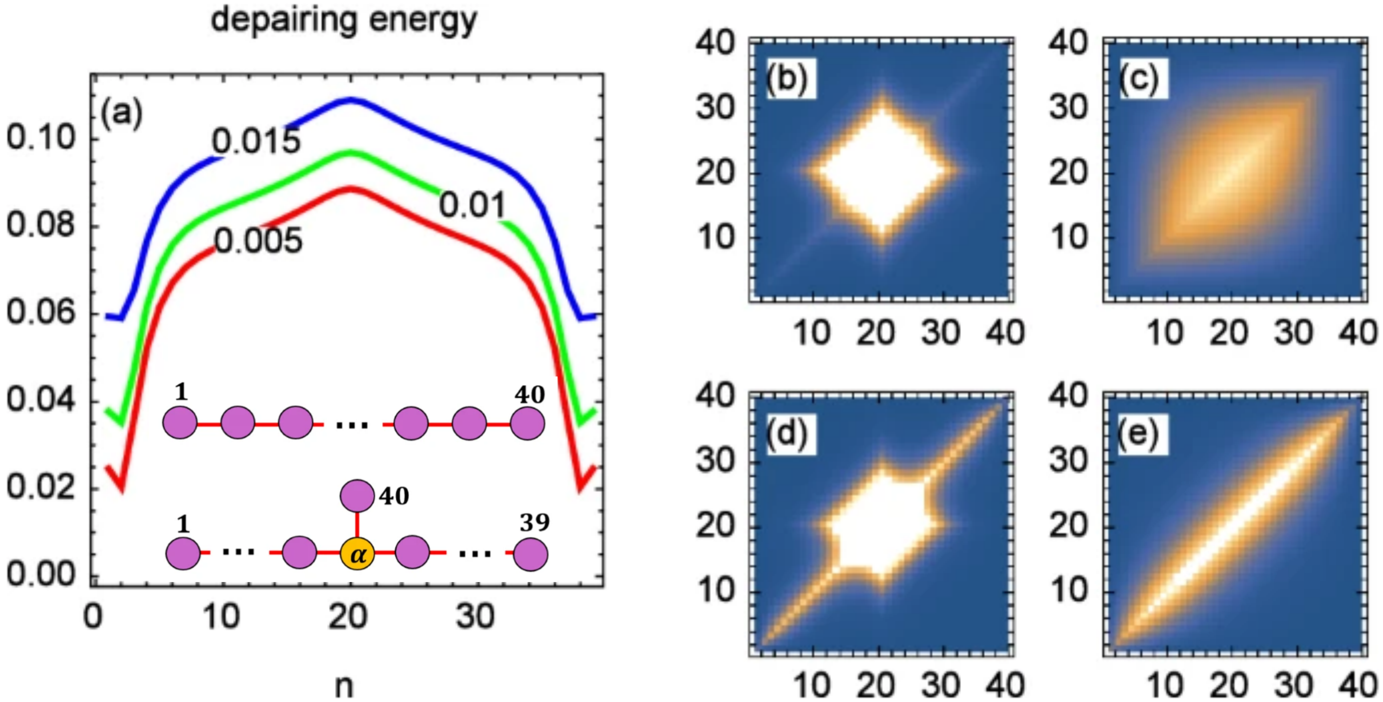
$$(\Delta_1 + \Delta_2)\phi(x_1, x_2) - g\delta_{x_1 x_2} \sum_y \phi(y, y) = E\phi(x_1, x_2)$$

It is easy to show that there is a single paired state with energy eigenvalue:

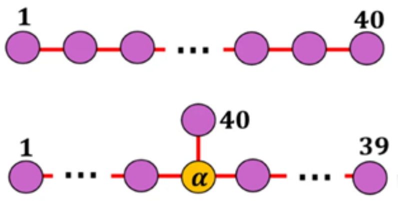
$$E_0 = -\sqrt{G^2 + 16K^2}$$

$$G = gN$$

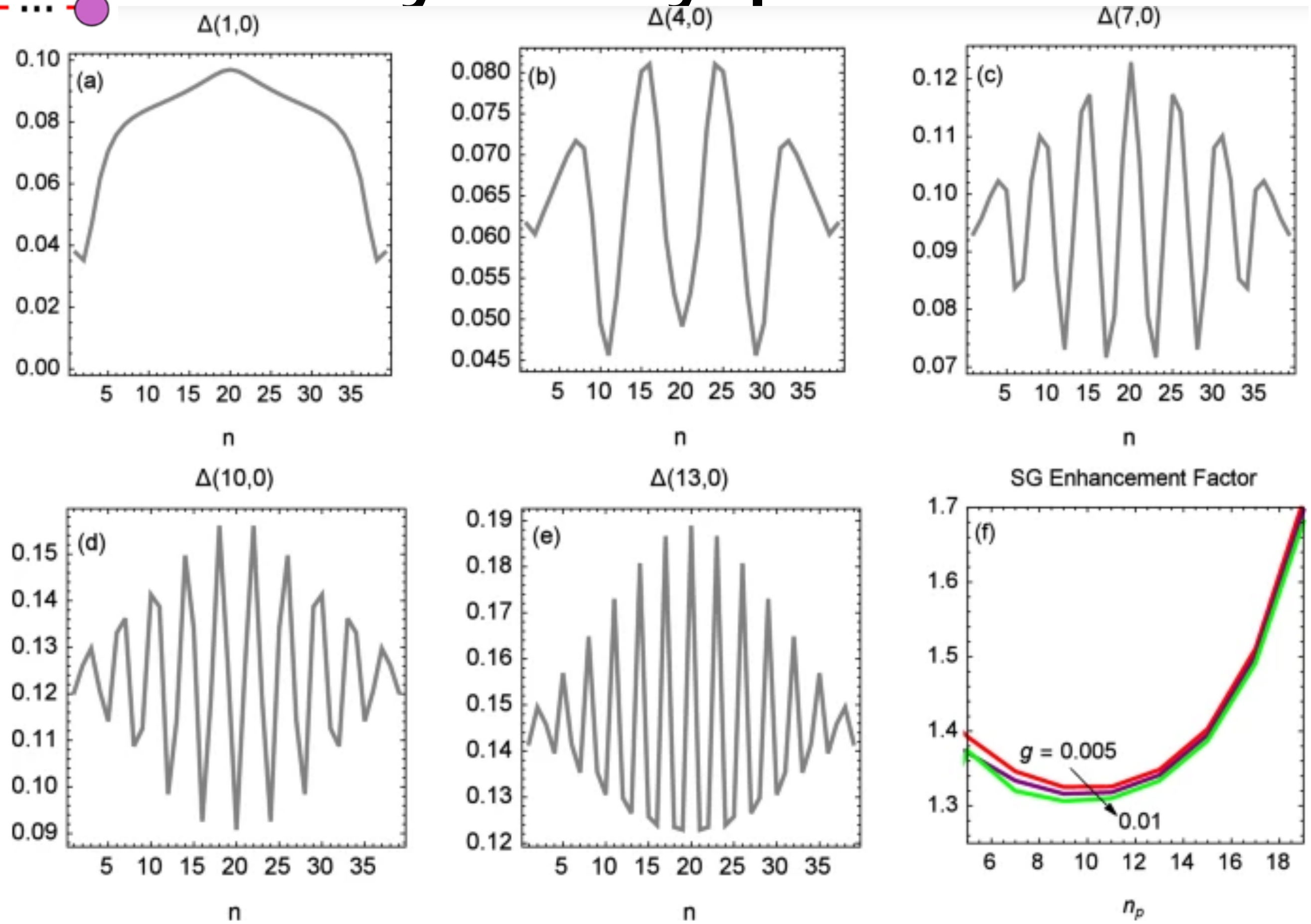
Two-body problem with rsBCS pairing: results for linear chains perturbed by a single lateral site



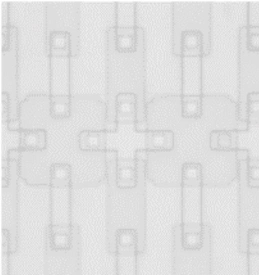
Panel **a** Depairing energy of the system depicted in Fig. 4b as a function of the position $n \in \{1, \ldots, 39\}$ of a single side site. Different curves are obtained by setting distinct values of the interaction, i.e., $g = 0.005$ (lower curve), $g = 0.01$ (middle curve) and $g = 0.015$ (upper curve). The depairing energy depends on the quantum graph topology and it is maximized when the side site is located at $n = 20$. The modulus squared of the ground-state wavefunction of the two-particle problem is shown in panels (**b**)–(**e**). In particular, panels **b** and **c** have been obtained by fixing $g = 0.005$ and $n = 20$ or $n = 1$, respectively. Panels **d** and **e** have been obtained by fixing $g = 0.015$ and $n = 20$ or $n = 1$, respectively. Interestingly, the most connected site acts as a localization center for the particles



Many-body problem



Curves of the spectral gap $\Delta(n_p, 0)$ as a function of the position n of the side site are reported in panels (**a**)–(**e**). Different panels are obtained by considering the network topology shown in Fig. 4b and setting $g = 0.01$ and $n_p \in \{1, 4, 7, 10, 13\}$, as specified in each panel. Panel **f** shows the spectral gap enhancement factor as a function of n_p. Different curves are obtained by considering distinct interaction values, i.e., $g = 0.005$ (upper curve), $g = 0.0075$ (middle curve) and $g = 0.01$ (lower curve)



Conclusions and …

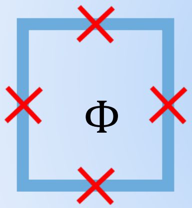
- ❑Effective interaction in quantum graphs can be controlled by the network connectivity
- ❑Connectivity acts as a confining potential: non-trivial interplay between many-body interaction and network topology
- ❑Possibility to obtain synthetic networks with critical temperature much higher than the islands critical temperature?
- ❑Relevance of the rsBCS model: exact solution of the many-body problem also for disordered systems, superlattice potentials, etc. Potential applications in cold atoms.
- ❑Open issues: e.g., quartet pairing in many-body theory? (partial answer in F. Romeo & A. Maiellaro, Bardeen-Cooper-Schrieffer interaction as an infinite-range Penson-Kolb pairing mechanism, PRB 109, 134509 (2024))

… perspectives

Is it possible to draw inspiration from the insights presented in the first part of the talk to *design qubit networks with topologies tailored for specific tasks*?

Understanding the Qubit Effective Hamiltonian

Qubit effective Hamiltonian



$$H_{eff} = -\epsilon(f)\,\sigma_z - \Delta\,\sigma_x$$
$$f = \Phi/\Phi_0$$
$$\epsilon(f) = I_S\,\Phi_0\,(f - 1/2)$$

https://www.nature.com/articles/srep28622

A two-level system

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & - \end{pmatrix}$$

$$\cos(\theta) = \epsilon/\sqrt{\epsilon^2 + \Delta^2}, \quad \sin(\theta) = \Delta/\sqrt{}$$

Solving the Eigenproblem

$$H_{eff} = -(\epsilon\sigma_z + \Delta\,\sigma_x) = -\sqrt{\epsilon^2 + \Delta^2}\,R(\theta)$$

$$H_{eff}\,|g\rangle = -\sqrt{\epsilon^2 + \Delta^2}\,|g\rangle$$
$$H_{eff}\,|e\rangle = +\sqrt{\epsilon^2 + \Delta^2}\,|e\rangle$$

$$R(\theta)|\pm\rangle = \pm\,|\pm\rangle$$

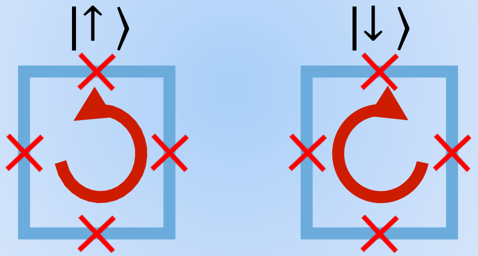
$$|+\rangle = \sqrt{(1+\cos(\theta))/2}\,|\uparrow\rangle + \sqrt{(1-\cos(\theta))/2}$$

$$|-\rangle = \sqrt{(1-\cos(\theta))/2}\,|\uparrow\rangle - \sqrt{(1+\cos(\theta))/2}$$

Current states

Current Operator & Current states

$I = -\partial H_{eff}/\partial\Phi = I_S\,\sigma$



$I\,|\uparrow\rangle = +I_S|\uparrow\rangle,\ |\downarrow\rangle = -I_S|\downarrow\rangle$

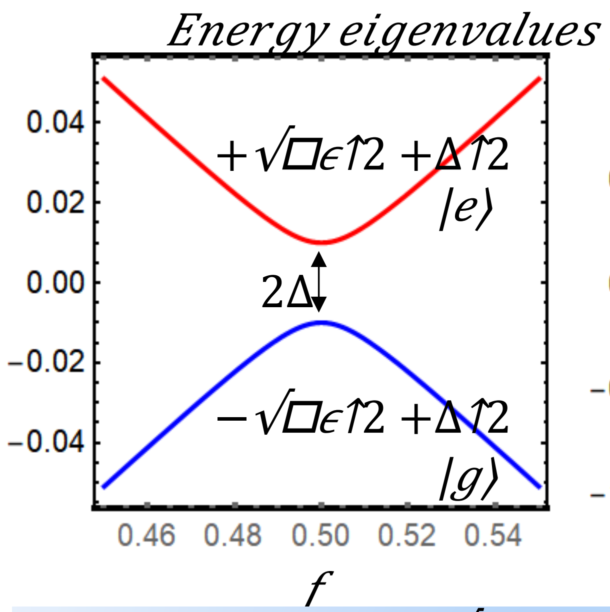
Coherent superposition of current states, e.g.

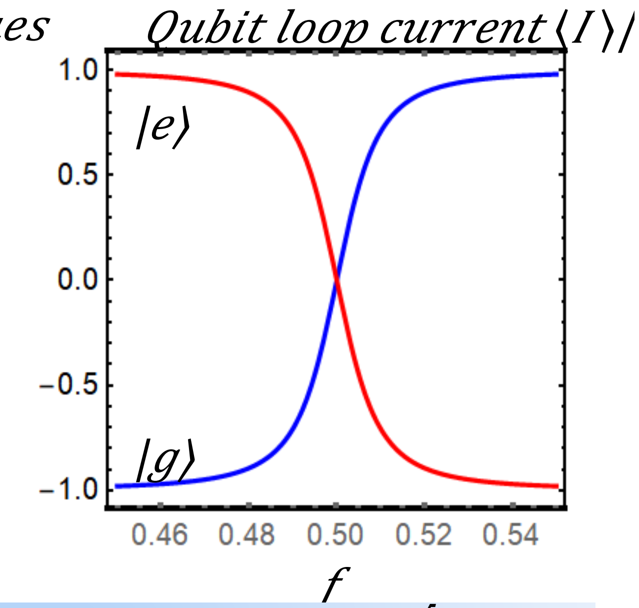
$|g; f=0\rangle = 1/\sqrt{2}\ \ (|\uparrow\rangle + |\downarrow\rangle)$

$|e; f=0\rangle = 1/\sqrt{2}\ \ (|\uparrow\rangle - |\downarrow\rangle)$

$E_{\pm} = \pm\sqrt{\epsilon^2 + \Delta^2}$

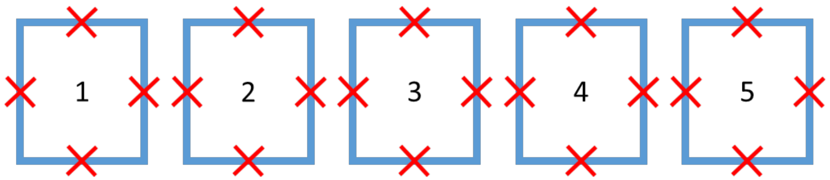
$\langle I\rangle_g = \langle g|I|g\rangle = +I_s$ co

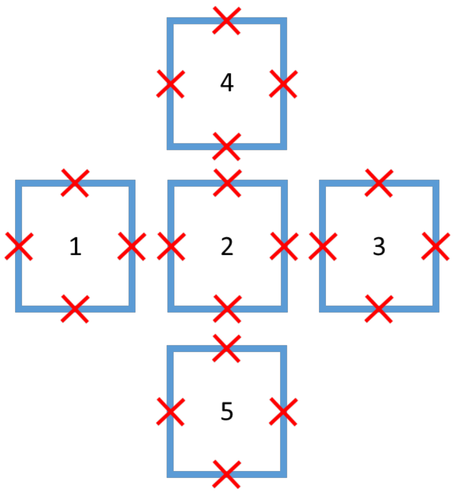




$\cos(\theta) = \epsilon/\sqrt{\epsilon^2+\Delta^2}\ \ = f - 1/2\ /\sqrt{(f-1/2}$

Modeling Qubits Arrays with inductive coupling: Network topology effects





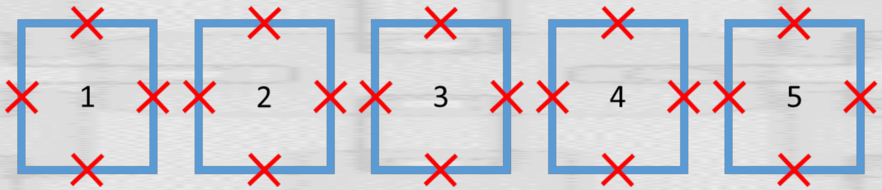
A Many-body Problem

$$H = \sum_{i} H_q^{(i)} + \frac{1}{2} \sum_{i,j;\, i \neq j} U_{ij}$$

$$H_q^{(i)} = -[\epsilon_i(f)\, \sigma_z^{(i)} + \Delta_i\, \sigma_x^{(i)}]$$

$$U_{ij} = \sum_{i,j;\, i \neq j} M_{ij}\, I_S^{(i)}\, I_S^{(j)}$$

Exact diagonalization: Linear Arrays



Assumptions

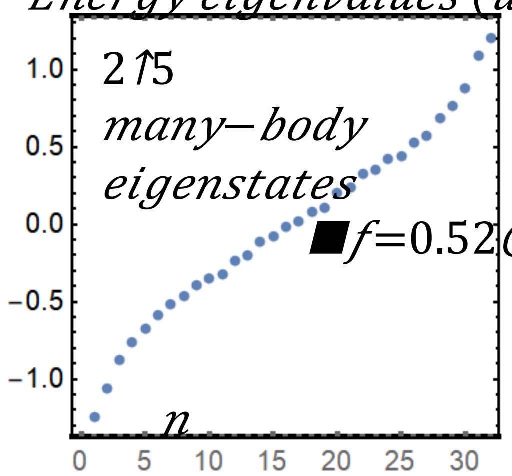
- "ferromagnetic" inductive coupling

  $M_{ij}<0$

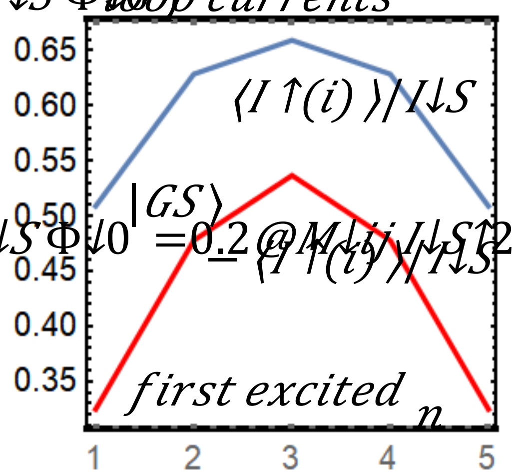
- nearest neighbors inductive coupling

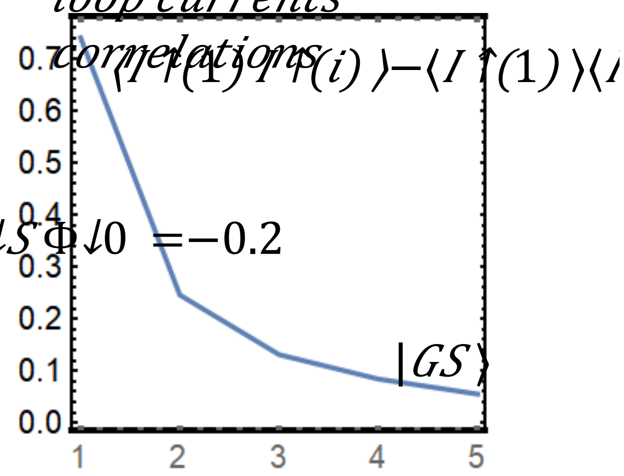
  $M_{ij}=0,\ if\ |i-j|>1$

- Absence of disorder

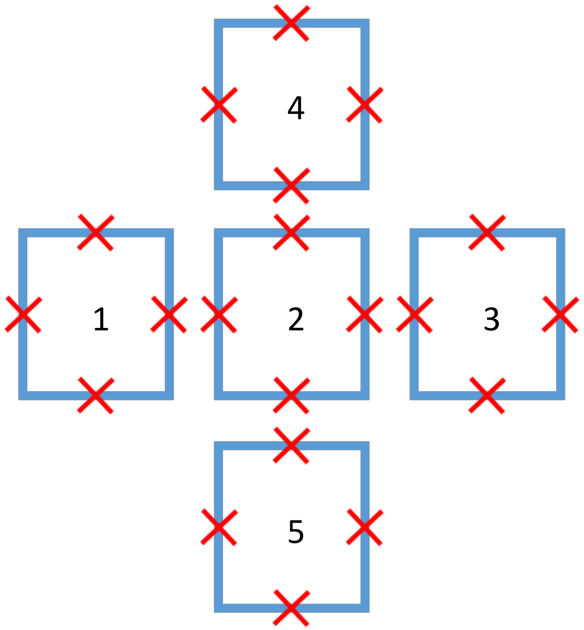


$f=0.52$, $\Delta/I_S\Phi_0=0.2$, $M_{ij}I_S^2/I_S\Phi_0=-0.2$

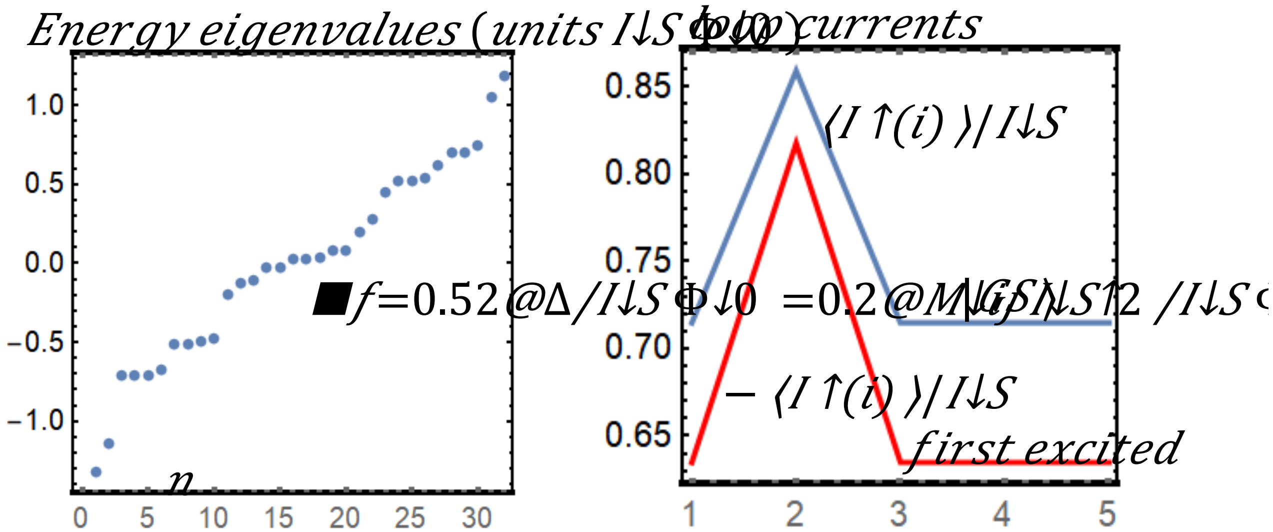




Exact diagonalization: Cross Arrays

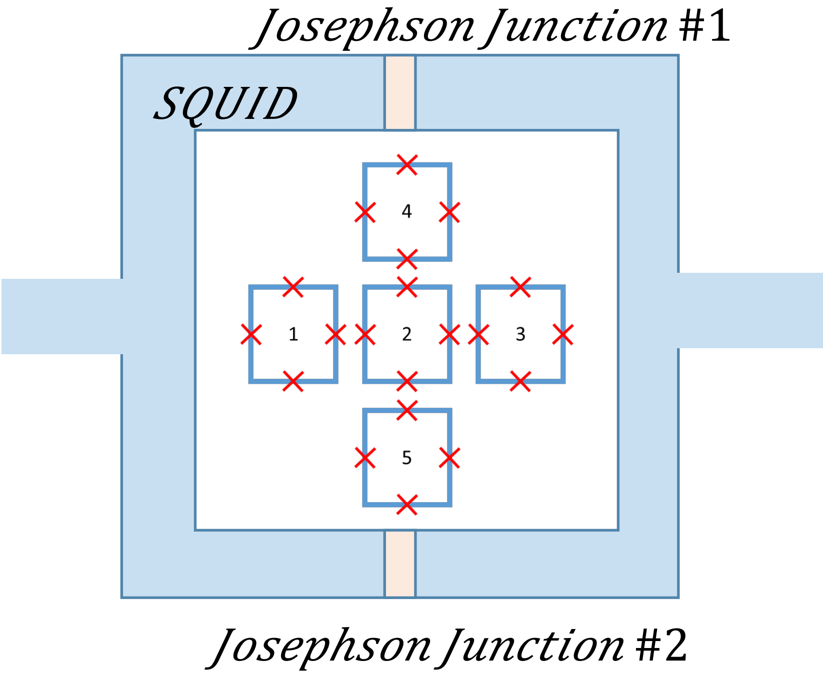


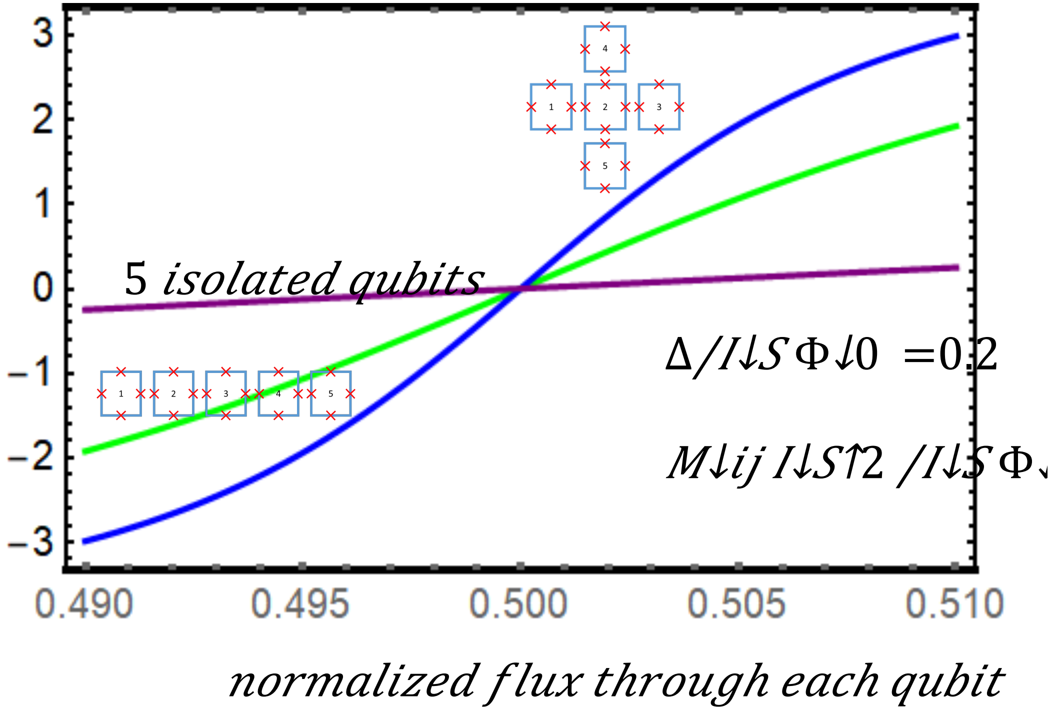
Topology-induced spectral properties engineering and loop currents enhancement

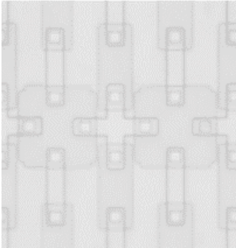


Detection

$$\text{generated flux} \propto \langle \sum_i \sigma_z^{(i)} \rangle_{GS}$$







An open-ended conclusion

- Can specific network topologies improve the robustness of quantum states against decoherence?
- Is there an optimal network topology for maximizing gate fidelity in quantum circuits?
- How does the dimensionality of a qubit network affect the scalability of quantum processors?
- What role can topology play in optimizing resource allocation for quantum sensing tasks?
- How does the interplay between physical qubit placement and logical connectivity affect computational efficiency?
- …